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# Borel approximation of coanalytic sets with Borel sections and the regularity properties for $\Sigma_2^1$ sets of reals

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In 2008, Fujita showed the following:

**Theorem 1** (Fujita [5]). The following are equivalent:

1. If  $A \subseteq \mathbb{R} \times \mathbb{R}$  is  $\Pi_1^1$  and for any real  $x$ ,  $A_x$  is Borel where  $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$ , then there is a comeager Borel set  $D \subseteq \mathbb{R}$  such that  $A \cap (D \times \mathbb{R})$  is Borel, and
2. every  $\Sigma_2^1$  set of reals has the Baire property.

We show that one can generalize the above theorem to a wide class of tree-type ccc forcings. More precisely:

**Theorem 2.** Let  $\mathbb{P}$  be a strongly arboreal,  $\Sigma_1^1$ , provably ccc forcing. Then the following are equivalent:

1. If  $A \subseteq \mathbb{R} \times \mathbb{R}$  is  $\Pi_1^1$  and for any real  $x$ ,  $A_x$  is Borel, then there is a Borel set  $D \subseteq \mathbb{R}$  such that  $D$  is of  $\mathbb{P}$ -measure one and  $A \cap (D \times \mathbb{R})$  is Borel, and
2. every  $\Sigma_2^1$  set of reals is  $\mathbb{P}$ -measurable.

We also show that this equivalence fails for non-ccc forcings. In fact, for Sacks forcing, the corresponding statement to the first fails in ZFC while the one for the second is consistent with ZFC.

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Throughout this paper, we assume the basic knowledge of descriptive set theory and forcing which can be obtained from e.g., [9], [8], and [2]. By *reals*, we mean elements of the Cantor space or those of the Baire space and we use  $\mathbb{R}$  to denote the set of reals.

To prove Theorem 2, let us start with basic definitions. Let  $n$  be a natural number with  $n \geq 1$ . A partial order  $\mathbb{P}$  is  $\Sigma_n^1$  if the sets  $P$ ,  $\leq_{\mathbb{P}}$ , and  $\perp_{\mathbb{P}}$  are  $\Sigma_n^1$ , where  $\mathbb{P} = (P, \leq_{\mathbb{P}})$  and  $\perp_{\mathbb{P}}$  is the incompatibility relation in  $\mathbb{P}$ . A partial order  $\mathbb{P}$  is *provably ccc* if there is a formula  $\phi$  defining  $\mathbb{P}$  and the statement “ $\phi$  defines a ccc partial order” is provable in ZFC. A partial order  $\mathbb{P}$  is *arboreal* if its conditions are perfect trees on  $\omega$  (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as  $\mathbb{P} = \{<^\omega \omega\}$ . We need the following stronger notion:

**Definition 3.** A partial order  $\mathbb{P}$  is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P}) (\forall t \in T) T_t \in \mathbb{P},$$

where  $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$ .

With strongly arboreal forcings, one can code generic objects by reals in the standard way: Let  $\mathbb{P}$  be strongly arboreal and  $G$  be  $\mathbb{P}$ -generic over  $V$ . Let  $x_G = \bigcup \{\text{stem}(T) \mid T \in G\}$ , where  $\text{stem}(T)$  is the longest  $t \in T$  such that  $T_t = T$ . Then  $x_G$  is a real and  $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$ , where  $[T]$  is the set of all infinite paths through  $T$ . Hence  $V[x_G] = V[G]$ . We call such real  $x_G$  a  *$\mathbb{P}$ -generic real over  $V$* .

Almost all typical forcings related to regularity properties are strongly arboreal:

**Example 4.**

1. Cohen forcing  $\mathbb{C}$ : Let  $T_0$  be  $<^\omega \omega$ . Consider the partial order  $(\{(T_0)_s \mid s \in <^\omega \omega\}, \subseteq)$ . Then this is strongly arboreal and equivalent to Cohen forcing.
2. Random forcing  $\mathbb{B}$ : Consider the set of all perfect trees  $T$  on 2 such that for any  $t \in T$ ,  $[T_t]$  has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.
3. Hechler forcing  $\mathbb{D}$ : For  $(n, f) \in \mathbb{D}$ , let

$$T_{(n,f)} = \left\{ t \in <^\omega \omega \mid \text{either } t \subseteq f \restriction n \text{ or } \left( t \supseteq f \restriction n \text{ and } (\forall m \in \text{dom}(t)) t(m) \geq f(m) \right) \right\}.$$

Then the partial order  $(\{T_{(n,f)} \mid (n,f) \in \mathbb{D}\}, \subseteq)$  is strongly arboreal and equivalent to Hechler forcing.

4. Mathias forcing  $\mathbb{R}_M$ : For a condition  $(s, A)$  in  $\mathbb{R}_M$ , let

$$T_{(s,A)} = \{t \in {}^{<\omega}\omega \mid t \text{ is strictly increasing and } s \subseteq \text{ran}(t) \subseteq s \cup A\}.$$

Then  $\{T_{(s,A)} \mid (s, A) \in \mathbb{R}_M\}$  is a strongly arboreal forcing equivalent to Mathias forcing.

5. Sacks forcing  $\mathbb{S}$ , Silver forcing  $\mathbb{V}$ , Miller forcing  $\mathbb{M}$ , Laver forcing  $\mathbb{L}$ : These forcings can be naturally seen as strongly arboreal forcings.

The following is as expected:

**Lemma 5.** Let  $\mathbb{P}$  be a strongly arboreal,  $\Sigma_1^1$ , provably ccc forcing and  $M$  be an inner model of ZFC containing parameter defining  $\mathbb{P}$  with a  $\Sigma_1^1$ -formula. Then if  $x$  is  $\mathbb{P}$ -generic over  $V$ , then  $x$  is  $\mathbb{P}^M$ -generic over  $M$ .

*Proof.* Since  $\mathbb{P}$  is  $\Sigma_1^1$ ,  $\mathbb{P}^M = \mathbb{P} \cap M$ . So it suffices to show that if  $A \subseteq \mathbb{P}^M$  is a maximal antichain in  $M$ , so is in  $V$ . Let  $A \subseteq \mathbb{P}^M$  be a maximal antichain in  $M$ . Since  $\mathbb{P}$  is provably ccc,  $M$  thinks  $\mathbb{P}^M$  is ccc. So  $A$  is countable in  $M$  and there is a real  $r$  coding  $A$  in  $M$ . Since  $\mathbb{P}$  is  $\Sigma_1^1$ , the statement “a real  $r$  codes a maximal antichain in  $\mathbb{P}$ ” is  $\Pi_2^1$ . So the real  $r$  also codes the maximal antichain  $A$  in  $V$ , as desired. ■

We now introduce a  $\sigma$ -ideal  $I_{\mathbb{P}}$  on the reals expressing “smallness” for each strongly arboreal forcing  $\mathbb{P}$ .

**Definition 6.** Let  $\mathbb{P}$  be a strongly arboreal forcing. A set of reals  $A$  is  $\mathbb{P}$ -null if for any  $T$  in  $\mathbb{P}$  there is a  $T' \leq T$  such that  $[T'] \cap A = \emptyset$ . Let  $N_{\mathbb{P}}$  denote the set of all  $\mathbb{P}$ -null sets and  $I_{\mathbb{P}}$  denote the  $\sigma$ -ideal generated by  $\mathbb{P}$ -null sets, i.e., the set of all countable unions of  $\mathbb{P}$ -null sets. A set of reals  $A$  is of  $\mathbb{P}$ -measure one if  $\mathbb{R} \setminus A$  is in  $I_{\mathbb{P}}$ .

**Example 7.**

1. Cohen forcing  $\mathbb{C}$ :  $\mathbb{C}$ -null sets are the same as nowhere dense sets of reals and  $I_{\mathbb{C}}$  is the ideal of meager sets of reals.

2. Random forcing  $\mathbb{B}$ :  $\mathbb{B}$ -null sets are the same as Lebesgue null sets in the Baire space and  $I_{\mathbb{B}}$  is the Lebesgue null ideal.

3. Hechler forcing  $\mathbb{D}$ :  $\mathbb{D}$ -null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by  $\{[s, f] \mid (s, f) \in \mathbb{D}\}$  where

$$[s, f] = \{x \in {}^\omega\omega \mid s \subseteq x \text{ and } (\forall n \geq \text{dom}(s)) x(n) \geq f(n)\}.$$

Hence  $I_{\mathbb{D}}$  is the meager ideal in the dominating topology.

4. Mathias forcing  $\mathbb{R}_M$ : A set of reals  $A$  is  $\mathbb{R}_M$ -null if and only if  $\{\text{ran}(x) \mid x \in A \cap A_0\}$  is Ramsey null or meager in the Ellentuck topology, where  $A_0$  is the set of strictly increasing infinite sequences of natural numbers. Hence  $I_{\mathbb{R}_M} = N_{\mathbb{R}_M}$ .

5. Sacks forcing  $\mathbb{S}$ : In this case,  $I_{\mathbb{S}} = N_{\mathbb{S}}$  by a standard fusion argument. The ideal  $I_{\mathbb{S}}$  is called the Marczewski ideal and often denoted by  $s_0$ .

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation  $I_{\mathbb{P}} = N_{\mathbb{P}}$ . In the case of ccc forcings,  $I_{\mathbb{P}}$  is often different from  $N_{\mathbb{P}}$  (e.g., Cohen forcing and Hechler forcing).

We now introduce  $\mathbb{P}$ -measurability:

**Definition 8.** Let  $\mathbb{P}$  be strongly arboreal. A set of reals  $A$  is  $\mathbb{P}$ -measurable if for any  $T$  in  $\mathbb{P}$  there is a  $T' \leq T$  such that either  $[T'] \cap A \in I_{\mathbb{P}}$  or  $[T'] \setminus A \in I_{\mathbb{P}}$ .

As is expected,  $\mathbb{P}$ -measurability coincides with a known regularity property for  $\mathbb{P}$  when  $\mathbb{P}$  is ccc:

**Proposition 9.** Let  $\mathbb{P}$  be a strongly arboreal, ccc forcing and let  $A$  be a set of reals. Then  $A$  is  $\mathbb{P}$ -measurable if and only if there is a Borel set  $B$  such that  $A \Delta B \in I_{\mathbb{P}}$ , where  $A \Delta B$  is the symmetric difference between  $A$  and  $B$ .

*Proof.* See Proposition 2.9 in [6]. □

Proposition 9 does not hold for non-ccc forcings such as Sacks forcing.<sup>1</sup>

But  $\mathbb{P}$ -measurability is almost the same as the regularity properties for non-ccc forcings  $\mathbb{P}$ , e.g., for Mathias forcing, a set of reals  $A$  is  $\mathbb{R}_M$ -measurable if and only if  $\{\text{ran}(x) \mid x \in A \cap A_0\}$  is completely Ramsey (or has the

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<sup>1</sup>For example, assuming every  $\Pi_1^1$  set has the perfect set property (i.e., either the set is countable or contains a perfect subset), there is no  $\Sigma_1^1$  Bernstein set (i.e., a set where neither it nor its complement contains a perfect subset) but for a  $\Sigma_1^1$  set of reals  $A$ ,  $A$  is approximated by a Borel set modulo  $I_{\mathbb{S}}$  if and only if  $A$  is Borel. This is because  $I_{\mathbb{S}}$  restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals by the assumption that every  $\Pi_1^1$  set has the perfect set property.

Baire property in the Ellentuck topology), where  $A_0$  is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

**Proposition 10** (Brendle, Löwe). Let  $\Gamma$  be a topologically reasonable point-class, i.e., it is a set of sets of reals closed under continuous preimages and any intersection between a set in  $\Gamma$  and a closed set of reals. Then every set in  $\Gamma$  is  $\mathbb{S}$ -measurable if and only if there is no Bernstein set in  $\Gamma$ .<sup>2</sup>

*Proof.* See [3, Lemma 2.1]. □

As expected, every  $\Sigma_1^1$  set of reals is  $\mathbb{P}$ -measurable:

**Theorem 11.** Let  $\mathbb{P}$  be a strongly arboreal, proper forcing. Then every  $\Sigma_1^1$  set of reals is  $\mathbb{P}$ -measurable.

*Proof.* It follows from the fact that every  $\Sigma_1^1$  set of reals is universally Baire, that every universally Baire set of reals is  $\mathbb{P}$ -Baire, and that every  $\mathbb{P}$ -Baire set of reals is  $\mathbb{P}$ -measurable. For the details, see [4] and Section 3 in [6]. □

We are now ready to state the theorem characterizing the regularity properties for  $\Sigma_2^1$  sets of reals in terms of the existence of many generic reals over  $L[r]$  for a real  $r$ , which we will use for the proof of Theorem 2:

**Theorem 12.** Let  $\mathbb{P}$  be a strongly arboreal,  $\Sigma_1^1$ , provably ccc forcing. Then the following are equivalent:

1. Every  $\Sigma_2^1$  set of reals is  $\mathbb{P}$ -measurable, and
2. for any real  $r$ , the set of  $\mathbb{P}$ -generic reals over  $L[r]$  is of  $\mathbb{P}$ -measure one.

*Proof.* See Definition 2.11, Lemma 2.13 (3), Definition 2.15, Proposition 2.17 (3), and Theorem 4.4 in [6]. □

We are now ready to prove Theorem 2:

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<sup>2</sup>In general, the property not being a Bernstein set does not imply  $\mathbb{S}$ -measurability while the converse is true. By using the axiom of choice, one can construct a set of reals which is not  $\mathbb{S}$ -measurable but is not a Bernstein set.

*Proof of Theorem 2.* The argument is exactly the same as the one in Theorem 1 in [5]. For the sake of completeness, we will give the proof.

We first show the implication from 1. to 2. Let  $P$  be a  $\Sigma_2^1$  set of reals. We will show that  $P$  is  $\mathbb{P}$ -measurable. Since  $P$  is  $\Sigma_2^1$ , there is a  $\Pi_1^1$  set  $A \subseteq \mathbb{R} \times \mathbb{R}$  such that  $P = \{x \in \mathbb{R} \mid (\exists y) (x, y) \in A\}$ . By Kondô's uniformization theorem, there is a  $\Pi_1^1$  function  $f: P \rightarrow \mathbb{R}$  uniformizing  $A$ . Then for any real  $x$ ,  $f_x = \{y \mid (x, y) \in f\} = \{f(x)\}$  is Borel, so by applying the assumption for  $f$ , there is a Borel set  $D$  of reals such that  $D$  is of  $\mathbb{P}$ -measure one and  $f \cap (D \times \mathbb{R})$  is Borel. Hence  $P \cap D = \{x \mid (\exists y) (x, y) \in f \cap (D \times \mathbb{R})\}$  is  $\Sigma_1^1$  and is  $\mathbb{P}$ -measurable by Theorem 11. So by Proposition 9, there is a Borel set  $B$  such that  $(P \cap D) \triangle B$  is in  $I_{\mathbb{P}}$ . Since  $D$  is of  $\mathbb{P}$ -measure one,  $P \triangle B$  is also in  $I_{\mathbb{P}}$ . Again by Proposition 9,  $P$  is  $\mathbb{P}$ -measurable, as desired.

We now show the implication from 2. to 1. Let WO be the set of reals coding a well-order on  $\omega$ . It is well-known that WO is a complete  $\Pi_1^1$  set of reals. For an element  $w$  of WO,  $|w|$  denotes the countable ordinal that  $w$  codes. We need the following notion and lemma for the proof:

**Definition 13.** Let  $r$  be a real. A set  $X \subseteq \mathbb{R} \times \omega_1$  is  $\Pi_2^1(r)$  in the codes if the set

$$\{(x, w) \in \mathbb{R} \times \mathbb{R} \mid w \in \text{WO and } (x, |w|) \in X\}$$

is  $\Pi_2^1(r)$ .

**Lemma 14.** Let  $r$  be a real and  $X \subseteq \mathbb{R} \times \omega_1$  be  $\Pi_2^1(r)$  in the codes. Suppose that for any real  $x$  there is a  $\xi < \omega_1$  such that  $(x, \xi) \in X$ . Then there is a countable ordinal  $\delta$  such that for any  $\mathbb{P}$ -generic real  $x$  over  $L[r]$ , there is a  $\xi < \delta$  such that  $(x, \xi) \in X$ .

*Proof of Lemma 14.* Since  $X$  is  $\Pi_2^1(r)$  in the codes, pick a  $\Pi_2^1$ -formula  $\phi(x, w, v)$  such that

$$(\forall x, w) (\phi(x, w, r) \iff w \in \text{WO and } (x, |w|) \in X).$$

Let  $\tilde{\phi}(x, \xi, r)$  be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in \text{WO}) |w| = \xi \rightarrow \phi(x, w, r).$$

Then  $\tilde{\phi}$  is absolute among all the transitive proper class models of ZFC in which  $\xi$  is countable.

For each  $\xi < \omega_1$ , let

$$X_\xi = \{T \in \mathbb{P} \mid (T, \mathbf{1}_{\mathbb{P}_\xi}) \Vdash_{\mathbb{P} \times \mathbb{P}_\xi} \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})\}^{L[r]},$$

where  $\mathbb{P}_\xi$  is  $\text{Coll}(\omega, \xi)$  and  $\dot{x}$  is a canonical  $\mathbb{P}$ -name for a generic real.

We show that  $\bigcup_{\xi < \omega_1} X_\xi$  is a dense subset of  $\mathbb{P}^{L(\mathbb{R})}$  in  $L[r]$ . Let  $T$  be any element of  $\mathbb{P}^{L[r]}$ . Take a  $\mathbb{P}$ -generic real  $x$  over  $L[r]$  in  $V$  with  $x \in [T]$ . Then by the assumption, there is a  $\xi < \omega_1$  such that  $(x, \xi) \in X$ . Take a function  $g: \omega \rightarrow \xi$  generic over  $L[r, x]$ . Then  $L[r, x, g] \models \tilde{\phi}(x, \xi, r)$ . Hence there is a  $T' \leq T$  and a condition  $p$  in  $\mathbb{P}_\xi$  such that  $L[r] \models "(T', p) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})"$ . Since  $\mathbb{P}_\xi$  is homogeneous, it follows that  $L[r] \models "(T', \mathbf{1}_{\mathbb{P}_\xi}) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})"$ , so  $T' \leq T$  and  $T \in \bigcup_{\xi < \omega_1} X_\xi$ , as desired.

Since  $\mathbb{P}$  is provably ccc,  $L[r] \models "\mathbb{P} \text{ is ccc}"$ , so there is a  $\delta < \omega_1$  such that  $\bigcup_{\xi < \delta} X_\xi$  is a predense subset of  $\mathbb{P}$  in  $L[r]$ . We show that this  $\delta$  is the desired countable ordinal. Take any  $\mathbb{P}$ -generic real  $x$  over  $L[r]$ . Then since  $L[r]$  thinks  $\bigcup_{\xi < \delta} X_\xi$  is a predense subset of  $\mathbb{P}$ , the generic filter  $G_x$  meets  $\bigcup_{\xi < \delta} X_\xi$  and hence there is a  $\xi < \delta$  such that  $G_x \cap X_\xi \neq \emptyset$ . By the definition of  $X_\xi$ , for a function  $g: \omega \rightarrow \xi$  generic over  $L[r, x]$ ,  $L[r, x, g] \models \tilde{\phi}(x, \xi, r)$ , hence  $\tilde{\phi}(x, \xi, r)$  holds also in  $V$  and  $(x, \xi) \in X$ , as desired.  $\square$  (Lemma 14)

We now finish showing the implication from 2. to 1. Let  $A \subseteq \mathbb{R} \times \mathbb{R}$  be  $\Pi_1^1$  such that for any real  $x$ ,  $A_x$  is Borel. Let  $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f^{-1}(\text{WO}) = A$ . Take any real  $x$ . Since  $A_x$  is Borel, the set  $f^{-1}(\{x\} \times A_x)$  is  $\Sigma_1^1$ , hence by boundedness theorem, it is bounded in WO, i.e.,

$$(\forall x) (\exists \xi) (\forall y) \text{ if } (x, y) \in A, \text{ then } |f(x, y)| < \xi.$$

Set

$$X = \{(x, \xi) \mid f^{-1}(\{x\} \times A_x) \subseteq \text{WO}_\xi\},$$

where  $\text{WO}_\xi = \{w \in \text{WO} \mid |w| < \xi\}$  for each  $\xi < \omega_1$ .

Then for any  $x$  there is a  $\xi$  with  $(x, \xi) \in X$ . It is also easy to see that  $X$  is  $\Pi_2^1(r)$  in the codes for some real  $r$ . By Lemma 14, there is a  $\delta < \omega_1$  such that for any  $\mathbb{P}$ -generic real  $x$  over  $L[r]$  there is a  $\xi < \delta$  such that  $(x, \xi) \in X$ . Hence  $A$  is the same as the Borel set  $f^{-1}(\text{WO}_\delta)$  on  $G(L[r]) \times \mathbb{R}$ , where  $G(L[r])$  is the set of  $\mathbb{P}$ -generic reals over  $L[r]$ . By 2. and Theorem 12, the set  $G(L[r])$  is of  $\mathbb{P}$ -measure one. Since  $\mathbb{P}$  is ccc,  $I_{\mathbb{P}}$  is Borel generated, so there is a Borel set  $D \subseteq G(L[r])$  of  $\mathbb{P}$ -measure one and  $A \cap (D \times \mathbb{R})$  is Borel, as desired.

■ (Theorem 2)



After the RIMS set theory conference in 2008, Fujita asked if one could take  $\delta$  in Lemma 14 below  $\gamma_2^1$  if  $X$  is  $\Pi_2^1$  (lightface) in the codes and if  $\mathbb{P}$  is Cohen forcing, where  $\gamma_2^1$  is the least countable ordinal that meets every set  $A \subseteq \omega_1$  which is  $\Pi_2^1$  (lightface) in the codes.<sup>3</sup> We show that this is generally the case for each strongly arboreal,  $\Sigma_1^1$ , provably ccc forcing  $\mathbb{P}$ :

**Proposition 15.** Let  $\mathbb{P}$  be a strongly arboreal,  $\Sigma_1^1$  (lightface), ccc forcing and  $X \subseteq \mathbb{R} \times \omega_1$  be  $\Pi_2^1$  (lightface) in the codes such that for any real  $x$  there is a  $\xi < \omega_1$  with  $(x, \xi) \in X$ . Then there is a  $\delta < \gamma_2^1$  such that for any  $\mathbb{P}$ -generic real  $x$  over  $L$ , there is a  $\xi < \delta$  with  $(x, \xi) \in X$ .

*Proof.* Let  $X \subseteq \mathbb{R} \times \omega_1$  be  $\Pi_2^1$  in the codes such that for any real  $x$  there is a  $\xi$  with  $(x, \xi) \in X$ .

Let  $A$  be as follows:

$$A = \{\gamma < \omega_1 \mid (\forall x: \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) (x, \xi) \in X\}.$$

By Lemma 14,  $A$  is nonempty. Hence it suffices to show that  $A$  is  $\Pi_2^1$  in the codes.

Since  $X$  is  $\Pi_2^1$  in the codes, pick a  $\Pi_2^1$ -formula  $\phi$  such that

$$(\forall x, w) (\phi(x, w) \iff w \in \text{WO and } (x, |w|) \in X).$$

Let  $\tilde{\phi}$  be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in \text{WO}) |w| = \xi \rightarrow \phi(x, w).$$

Then

$$\begin{aligned} A &= \{\gamma < \omega_1 \mid (\forall x: \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) \tilde{\phi}(x, \xi)\} \\ &= \{\gamma < \omega_1 \mid L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)" \}, \end{aligned}$$

where  $\mathbb{P}_\gamma$  is  $\text{Coll}(\omega, \gamma)$  and  $\dot{x}$  is a canonical  $\mathbb{P}$ -name for a generic real.

**Claim 16.** For  $\gamma < \omega_1$ ,

$$L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)" \iff V \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)"$$

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<sup>3</sup> $\gamma_2^1$  is also the least ordinal such that every  $\Pi_1^1$  (lightface) Borel set is  $\Pi_\alpha^0$  (boldface) for some  $\alpha < \omega_1$ . For the details, see [7].

*Proof of Claim 16.* The direction from left to right follows from the fact that if  $(x, g)$  is  $\mathbb{P} \times \mathbb{P}_\gamma$ -generic over  $V$ , then so is over  $L$  by Lemma 5.

For right to left, suppose  $L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \check{\phi}(\dot{x}, \xi)"$  fails. Then there is a  $(p, q) \in \mathbb{P} \times \mathbb{P}_\gamma$  in  $L$  such that  $L \models "(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\forall \xi < \check{\gamma}) \neg \check{\phi}(\dot{x}, \xi)"$ . Take a  $\mathbb{P} \times \mathbb{P}_\gamma$ -generic  $(x, g)$  over  $V$  with  $x \in [p]$  and  $g \supseteq q$ . By the assumption, there exists a  $\xi < \gamma$  such that  $V[x, g] \models \phi(x, \xi)$ . But  $(x, g)$  is also  $\mathbb{P} \times \mathbb{P}_\gamma$ -generic over  $L$  and  $L[x, g] \models \check{\phi}(x, \xi)$ , contradicting  $L \models "(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\forall \xi < \check{\gamma}) \neg \check{\phi}(\dot{x}, \xi)"$ .  $\square$  (Claim 16)

Therefore,

$$A = \{\gamma < \omega_1 \mid "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \check{\phi}(\dot{x}, \xi)"\}.$$

Let  $\psi$  be the following:

$$\psi(w) \iff w \in \text{WO} \text{ and } (1_{\mathbb{P}}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \restriction n)",$$

where  $w \restriction n$  is the real coding the well-order  $\leq_w$  below  $n$ , i.e.  $\leq_{w \restriction n} = \{(l, m) \mid l \leq_w m <_w n\}$ . Then

$$(\forall w)(\psi(w) \iff w \in \text{WO} \text{ and } |w| \in A).$$

Hence it suffices to show that  $\psi$  is equivalent to a  $\Pi_2^1$ -formula. Since  $\mathbb{P}_{|w|}$  is ccc in  $V^{\mathbb{P}}$ ,  $\mathbb{P} \times \mathbb{P}_{|w|}$  is also ccc. Moreover, it is easy to see that  $\mathbb{P} \times \mathbb{P}_{|w|}$  is  $\Sigma_1^1(w)$  uniformly in  $w \in \text{WO}$ . Hence, by the same argument as in Theorem 2.7 (1) in Bagaria and Bosch [1], since  $(\exists n \in \omega) \phi(x, w \restriction n)$  is  $\Pi_2^1$  in  $x$  and  $w$ , so is  $(1_{\mathbb{P}}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \restriction n)"$  in  $w$ . Therefore,  $\psi$  is equivalent to a  $\Pi_2^1$ -formula.  $\blacksquare$  (Proposition 15)

As announced in the beginning of this paper, we now show that the first item in Theorem 2 fails in ZFC for  $\mathbb{P} = \mathbb{S}$  (Sacks forcing):

**Proposition 17.** There is a  $\Pi_1^1$  set  $A \subseteq \mathbb{R} \times \mathbb{R}$  such that for every  $x$ ,  $A_x$  is Borel and there is no set  $D$  of  $\mathbb{S}$ -measure one such that  $A \cap (D \times \mathbb{R})$  is Borel.

*Proof.* Let  $A$  be the following:

$$A = \{(x, y) \mid x, y \in \text{WO} \text{ and } |x| = |y|\}.$$

It is easy to see that  $A$  is  $\Pi_1^1$  and  $A_x$  is Borel for every  $x$ .

To derive a contradiction, let  $D$  be a set of  $\mathbb{S}$ -measure one such that  $A \cap (D \times \mathbb{R})$  is Borel. Let  $B$  be the projection of  $A \cap (D \times \mathbb{R})$  to the first coordinate. Then  $B$  is analytic and by boundedness lemma, there is a  $\delta < \omega_1$  such that the length of any element of  $B$  is less than  $\delta$ .

But this means that the set  $C = \{y \mid |y| = \delta\}$  is disjoint from  $B$ . Since  $C$  is a subset of the projection of  $A$  to the first coordinate, it is disjoint from  $D$  and it clearly contains a perfect set, contradicting the choice of  $D$ . ■

It is also notable that Lemma 14 can consistently fail for Sacks forcing:

**Proposition 18.** Let  $s$  be a Sacks real over  $L$ . Then in  $L[s]$ , there is an  $X \subseteq \mathbb{R} \times \omega_1$  which is  $\Pi_2^1$  in the codes such that for every real  $x$ , there is a  $\xi < \omega_1$  with  $(x, \xi) \in X$  and that there is no  $\delta < \omega_1$  such that for any Sacks real  $x$  over  $L$ , there is a  $\xi < \delta$  with  $(x, \xi) \in X$ .

*Proof.* We work in  $L[s]$ . Let  $X$  be the following:

$$X = \{(x, \xi) \mid x \in \text{WO and } |x| = \xi\} \cup \{(x, 0) \mid x \notin \text{WO}\}.$$

It is easy to see that  $A$  is  $\Pi_2^1$  in the codes and that for any every  $x$  there is an ordinal  $\xi$  with  $(x, \xi) \in A$ .

To derive a contradiction, suppose there is a  $\delta < \omega_1$  such that for any Sacks real  $x$  over  $L$ , there is a  $\xi < \delta$  with  $(x, \xi) \in A$ . It is easy to find a non-constructible surjection from  $\omega$  to  $\delta$ . Code that real as a relation on  $\omega$  and make it a real in WO. Call it  $x$ . Then  $(x, \delta) \in A$ . But since  $x$  is non-constructible,  $x$  is also a Sacks real over  $L$ , contradicting the choice of  $\delta$ . ■

Finally note that the second item of Theorem 2 for Sacks forcing is consistent with ZFC: In fact, it is equivalent to the statement that for any real  $r$  there is a real  $x$  which is not in  $L[r]$ ,<sup>4</sup> which is easily seen to be consistent with ZFC.

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<sup>4</sup>For the proof, see [3, Theorem 7.1].

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